

# Temperature of a Decoherent Oscillator with Strong Coupling

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The temperature of an oscillator coupled to the vacuum state of a heat bath via ohmic coupling is non-zero, as measured by the reduced density matrix of the oscillator. This paper shows that the actual temperature, as measured by a thermometer is still zero (or in the thermal state of the bath, the temperature of the bath). The decoherence temperature is due to "false-decoherence", with the heat bath state being dragged along with the oscillator.

There are at least two ways of defining the temperature of a system. Assuming that the system is in equilibrium, one definition would be to look at the reduced density matrix of the system under consideration. If the system is in thermal equilibrium one would expect that reduced density matrix to have the form

$$\rho = N e^{-\frac{H}{T}} \quad (1)$$

where  $N$  is a normalisation factor such that  $Tr(\rho) = 1$ ,  $H$  is an (effective) Hamiltonian for the system, and  $T$  is the temperature of the system.

An alternative way of defining the temperature is that one measures it with a thermometer. One (weakly) couples a system to the object under consideration, and looks at some physical feature of that system which depends on its temperature.

One of the key features of a thermal density matrix is that the density matrix is not that of a pure state. It is a mixed state, in which there are no correlations between the various eigenstates of the Hamiltonian  $H$ . Another feature of the reduced density matrix is that it is "instantaneous"—one calculates the reduced density matrix at some instant in time.

In a previous paper, I defined the concept of "false decoherence" [1]. This is a situation in which the decoherence of the density matrix of a system could disappear under certain circumstances. One can make certain measurements on the system which would show the state of the system to be that of a pure state. In that case, the instantaneous measurement of the system was such that at one time, the density matrix of the system was such that it was almost completely decohered (for a two level system, was the identity matrix) while at a later time the density matrix again looked like an almost pure state. This situation differs from quantum recurrence, where the dynamics of the heat bath which cause the decoherence recur (as in spin echo). Instead the false decoherence occurs because the decoherence is essentially the result of correlations between the object and the heat bath which remain localized around the object, and depend on the state of the object. Thus when one traced out the state over the heat bath, one could find that that tracing could at one time produce a decoherent state for the system itself, while at a later time, that tracing would leave the system in a pure state.

A hand-waving example of this is to regard an electron taking two different paths through an electron microscope. If one traces out over the degrees of freedom of the electromagnetic field, one finds that the Coulomb field of the electron decoheres the state of the electron. The overlap integral of the Coulomb fields of an electron at distant points in space goes to zero, and the state of the electron decoheres if widely separated. But if the electrons are then re-guided back to a common point, the electron having travelled along different paths will again interfere, since the Coulomb field of the electron located at nearby positions will again overlap, and the decoherence which arose out of that lack of overlap will again disappear. (Note that this differs from the situation in which the electron radiates differently along its different possible paths. In that case the electromagnetic field will cause genuine, long lasting, loss of coherence).

In this paper I will look at another instance of such false decoherence. In this case the object of interest will be a simple Harmonic oscillator coupled to a one dimensional scalar field as a heat bath. The scalar field will be taken to be a field in a vacuum state. Because of the strong coupling of the field to the oscillator, the reduced density matrix of the oscillator will not be in its ground state, but rather in a thermal state with some effective Hamiltonian defining that thermal state. I will model the thermometer as another harmonic oscillator very weakly coupled to the first oscillator, to minimize the impact of that second oscillator on the first. In all cases I will wait long enough that any

transients will have died away. I will demonstrate that just as in the cases discussed previously, this system also shows an aspect of false decoherence. The density matrix of the first oscillator will be shown to be in a thermal state with a possibly large temperature, and a large entropy. Nevertheless, the thermometer, i.e., the second oscillator which couples only weakly to the first oscillator, will be shown to be in its ground state, a completely coherent state. It measures the temperature of a system with a thermal density matrix as a completely coherent state.

Note that a one dimensional massless scalar field radiates extremely easily. There are no phase space barriers to radiation (the volume of phase space is the same at all energies). Thus one would expect any system coupled to a massless two-dimensional scalar field to decoher very easily. Nevertheless, we shall find that, while the coupling of the oscillator to the scalar field in its vacuum state indeed does decoher the oscillator, there are measurements one can make on the oscillator which show it to be in a coherent state. In particular the thermometer attached to the oscillator shows it to have zero temperature.

## I. STRONGLY COUPLED OSCILLATOR

Let us consider an oscillator, with dynamic degree of freedom  $q$  coupled to a heat bath represented by a scalar one dimensional field  $\phi$  and the oscillator located at position  $x = 0$  in the space of the scalar field. The total action is given by

$$I = \frac{1}{2}(\dot{q}^2 - \Omega^2 q^2) + \frac{1}{2} \int (\dot{\phi}^2 - \phi'^2) + 2\epsilon q \dot{\phi}(t, 0) \quad (2)$$

where  $\dot{\phantom{x}}$  denotes the time derivative and  $'$  the spatial ( $x$ ) derivative. The equation of motion for the field is

$$\ddot{\phi} - \phi'' = -\epsilon q \delta(x) \quad (3)$$

which has the solution

$$\phi = \phi_0(t, x) - \frac{\epsilon}{2} q(t - |x|) \quad (4)$$

where  $\phi_0$  is the “incoming” free field.

Substituting this into the equation of motion for  $q$  gives

$$\ddot{q} + \frac{\epsilon^2}{2} \dot{q} + \Omega^2 q = \epsilon \dot{\phi}_0(t, 0) \quad (5)$$

This model thus has the standard Ohmic damping for the oscillator. Writing  $\phi_0$  in terms of the right and left movers  $\phi_0(t, x) = \phi_+(t - x) + \phi_-(t + x)$ , we have (with  $q(t) = \int q(\omega) \frac{e^{i\omega t}}{\sqrt{2\pi}}$ )

$$q(\omega) = \frac{i\epsilon\omega}{-\omega^2 + i\frac{\epsilon^2}{2}\omega + \Omega^2} (\phi_+(\omega) + \phi_-(\omega)) \quad (6)$$

The field operators  $\phi_{\pm}(\omega) = \int \phi_{\pm}(t) e^{-i\omega t} \frac{dt}{\sqrt{2\pi}}$  are creation/annihilation operators such that

$$[\phi_{\alpha}(\omega), \phi_{\beta}(\omega')] = \frac{1}{2\omega} \delta_{\alpha\beta} \delta(\omega + \omega') \quad (7)$$

and the vacuum state for the incoming scalar field is given by

$$\phi_{\pm}(-|\omega|)|0\rangle = 0 \quad (8)$$

which gives

$$\langle \phi_{\alpha}(\omega) \phi_{\beta}(\omega') \rangle = -\frac{1}{2\omega} \delta_{\alpha\beta} \delta(\omega + \omega') \Theta(-\omega) \quad (9)$$

where  $\alpha, \beta$  have values  $\pm$ , and  $\Theta$  is the Heaviside step function.

We are interested in the correlations  $\langle q^2 \rangle$ ,  $\langle pq + qp \rangle$ , and  $\langle p^2 \rangle$  in the limit as  $t$  gets very large so that any initial conditions of the oscillator have died out.

We have

$$\langle q^2 \rangle = \int_0^\infty \frac{\epsilon^2}{2\pi} \frac{\omega^2}{(\omega^2 - \Omega^2)^2 + (\frac{\epsilon^2 \omega}{2})^2} d\omega \quad (10)$$

$$\langle qp + pq \rangle = 0 \quad (11)$$

$$\langle p^2 \rangle = \int_0^\infty \frac{\epsilon^2}{2\pi} \frac{\omega^3}{(\omega^2 - \Omega^2)^2 + (\frac{\epsilon^2 \omega}{2})^2} d\omega \quad (12)$$

Note that for  $\langle p^2 \rangle$ , the integral diverges logarithmically at the upper end, and requires a cutoff to keep it finite. This cutoff could be introduced in the interaction between the oscillator and the field (e.g., by replacing the delta function interaction between the field and the oscillator with some sharply peaked function  $\int \dot{\phi}(t, x)q(t)dx$ , or it could be introduced in the interaction between the oscillator and any measuring apparatus which attempts to measure its momentum: if that measurement had some finite duration, that would also provide an effective cutoff. I will just take a simple cutoff frequency  $\hat{\Omega}$  to the integral, with  $\epsilon^2 \ll \hat{\Omega} \ll \Omega$ .

We can also write the two time correlation function for  $q$  as

$$\langle q(t)q(t') + q(t')q(t) \rangle = \int_0^\infty \frac{2\epsilon^2}{2\pi} \frac{\omega}{(\omega^2 - \Omega^2)^2 + (\frac{\epsilon^2 \omega}{2})^2} \cos(\omega(t - t')) d\omega \quad (13)$$

in which case

$$\langle qp + pq \rangle = \frac{d \langle q(t)q(t') + q(t')q(t) \rangle}{dt} \Big|_{t'=t} = 0 \quad (14)$$

because the correlation function is symmetric in  $t - t'$  and its derivative is antisymmetric. Furthermore,

$$2 \langle p^2 \rangle = \frac{d^2 \langle q(t)q(t') + q(t')q(t) \rangle}{dt dt'} \Big|_{t'=t} = \langle -\frac{d^2 q}{dt^2} q(t') - q(t') \frac{d^2 q}{dt^2} \rangle \Big|_{t'=t} \quad (15)$$

$$= \frac{\epsilon^2}{2} \langle pq + qp \rangle + \Omega^2 \langle q^2 \rangle + \epsilon \langle \phi_0 q + q \phi_0 \rangle \quad (16)$$

The last term is proportional to  $\epsilon^2$  and is the term which gives the logarithmic divergence in  $\langle p^2 \rangle$ .

The expression  $\langle p^2 \rangle \langle q^2 \rangle - \frac{1}{4} \langle pq + qp \rangle^2$  is invariant under any linear symplectic transformations of  $p$  and  $q$ , (i.e., a transformation which leaves the form  $p\dot{q}$  invariant up to a complete time derivative). These can be made up of successive applications of the transformations of the form,  $\tilde{p} = \alpha p$ ,  $\tilde{q} = \frac{1}{\alpha} q$ ;  $\tilde{p} = p + \alpha q$ ,  $\tilde{q} = q$ ; or  $\tilde{p} = p$ ,  $\tilde{q} = q + \alpha p$ .

The reduced density matrix for the oscillator after integrating over the fields will be a quadratic Gaussian form, so that

$$\rho = N e^{-\tilde{H}/T} \quad (17)$$

for some quadratic form  $\tilde{H}$  and some temperature  $T$ .  $N$  is a normalisation constant, so that  $Tr(\rho) = 1$ . (If  $T \rightarrow 0$ , it will be a Gaussian pure state). One can always make a symplectic transformation so that any quadratic form of the  $p$  and  $q$  has the form

$$\tilde{H} = \frac{1}{2} \Lambda (\tilde{p}^2 + \tilde{q}^2) \quad (18)$$

for which a 'thermal state'  $\rho = N e^{-\tilde{H}/T}$  has the properties that

$$\langle \tilde{p}^2 \rangle = \langle \tilde{q}^2 \rangle = \frac{1}{2} \coth \left( \frac{\Lambda}{2T} \right) \quad (19)$$

$$\langle \tilde{p}\tilde{q} + \tilde{q}\tilde{p} \rangle = 0 \quad (20)$$

Thus,

$$\langle p^2 \rangle \langle q^2 \rangle - \frac{1}{4} \langle pq + qp \rangle^2 = \frac{1}{4} \coth^2 \left( \frac{\Lambda}{2T} \right) \quad (21)$$

So, unless  $\langle p^2 \rangle \langle q^2 \rangle - \frac{1}{4} \langle pq + qp \rangle^2 = \frac{1}{4}$ , the reduced density matrix of the oscillator is a thermal state with a non-zero entropy and temperature. Only in the limit as  $\epsilon$  goes to 0 is this effective temperature equal to 0.

Evaluating  $\langle q^2 \rangle$  and ignoring terms which go as inversely as the cutoff, we have for  $\epsilon^2 < 4\Omega$

$$\langle q^2 \rangle = \int \frac{\epsilon^2}{\pi} \frac{\omega}{(\omega^2 - \Omega^2)^2 + \omega^2 \frac{\epsilon^4}{4}} d\omega \quad (22)$$

$$= \frac{\epsilon^2}{2\pi} \int \frac{dy}{(y - \Omega^2)^2 + y \frac{\epsilon^4}{4}} \quad (23)$$

$$= \frac{\epsilon^2}{2\pi} \frac{1}{\sqrt{\epsilon^4 \Omega^2 - \epsilon^8/16}} \left[ 2\pi - 2 \operatorname{atan} \left( \frac{2\epsilon^2 \sqrt{16\Omega^2 - \epsilon^4}}{8\Omega^2 - \epsilon^4} \right) \right] \quad (24)$$

For small  $\epsilon$  we get

$$\langle q^2 \rangle = \frac{1}{2\Omega} + O(\epsilon^4) \quad (25)$$

If  $\epsilon^2 > 4\Omega$ , we have

$$\langle q^2 \rangle = \frac{1}{\sqrt{\epsilon^2/16 - \Omega^2}} \ln \left( \frac{\epsilon^4 - 16\Omega^2 + 2\epsilon^2 \sqrt{\epsilon^4 - 16\Omega^2}}{\epsilon^4 - 16\Omega^2 - 2\epsilon^2 \sqrt{\epsilon^4 - 16\Omega^2}} \right) \quad (26)$$

For  $\langle p^2 \rangle$  we have

$$\langle p^2 \rangle = (2\Omega^2 - \epsilon^4/4) \langle q^2 \rangle + \frac{2\epsilon^2}{\pi} \ln \left( \frac{\dot{\Omega}}{\Omega} \right) \quad (27)$$

Similarly

$$\langle p^2 \rangle = \Omega^2 \langle q^2 \rangle + \frac{\omega^2 \epsilon^2}{(\omega^2 - \Omega^2) + i\omega \epsilon^2/2} \dots \quad (28)$$

## II. TEMPERATURE VIA THERMOMETRY

The density matrix of the oscillator looks very much like a thermal state notwithstanding its Hamiltonian which is not exactly the same as the original uncoupled Hamiltonian. But is this really a temperature? To answer this question we can try to couple a thermometer to the oscillator. In this case my thermometer will be another harmonic oscillator weakly coupled to the original one. We know that in the limit as the coupling strength goes to 0, the temperature of the oscillator is the temperature of the system it is coupled to.

Thus we take the Lagrangian density as

$$L = \int \left( \frac{1}{2} (\dot{\phi}^2 - \phi'^2) + \epsilon \dot{\phi} q \delta(x) \right) dx + \frac{1}{2} (\dot{q}^2 - \Omega^2 q^2) + \mu q z + \frac{1}{2} (z^2 + \Lambda^2 z^2) \quad (29)$$

The solution for the thermometer is

$$z(\omega) = -\mu \epsilon \frac{i\omega}{(-\omega^2 + \Lambda^2)(-\omega^2 + i\omega \epsilon^2/2 + \Omega^2) - \mu^2} (\phi_+(\omega) + \phi_-(\omega)) \quad (30)$$

and

$$\langle z(-\omega) z(\omega) \rangle = \frac{2}{2\pi} \mu^2 \epsilon^2 \frac{\omega}{((- \omega^2 + \Lambda^2)(-\omega^2 + \Omega^2) - \mu^2)^2 + (-\omega^2 + \Lambda^2)^2 \omega^2 \epsilon^4/4} \quad (31)$$

We want the limit of this as  $\mu \rightarrow 0$ . In that limit, the only terms that will survive in the integral to find  $\langle z^2 \rangle = \int_0^\infty \langle z(-\omega) z(\omega) \rangle d\omega$  are those due to the poles at

$$\omega \approx \Lambda + \frac{\mu^2}{2\Lambda(\Lambda^2 - \Omega^2 \pm i\frac{\epsilon^2}{2})} \quad (32)$$

which in the limit as  $\mu \rightarrow 0$  give

$$\langle z^2 \rangle = \frac{1}{2\Lambda} + O(\mu^2) \quad (33)$$

Similarly, the momentum is

$$\langle p_z^2 \rangle = \frac{\Lambda}{2} + O(\mu^2) \quad (34)$$

and  $\langle p_z z + z p_z \rangle = 0$ . In the limit as  $\mu \rightarrow 0$  and at long enough times such that the thermometer has come into equilibrium, the thermometer is in its ground state, with a temperature of 0 (to order  $\mu^2$ ).

The  $q$  oscillator is, at all times, in a time independent mixed state. The thermometer interacts only with that system in a mixed state, and one would thus expect it also to be in a mixed state. However, the thermometer at long times ends up in a pure state, with a temperature of 0.

Of course, if one regards the thermometer as interacting with the complete system, the  $q$  oscillator and its heat bath, then, after a long time one expects that system to be in its ground state, with temperature 0, and thus the thermometer would also be expected to read 0, as it does.

This is another example of "false decoherence". As in that previous work, the apparent decoherence of the spin system did not preclude interference between the "decohered" states of the spin, as long as the process revealing that interference occurred slowly enough, here again, the thermometer interacting slowly enough with the  $q$  oscillator displays interference between the various occupied energy levels of that oscillator, leaving the thermometer in its ground state.

This also suggests that just as in the spin case, the "true" states of the oscillator in the adiabatic limit are not the energy eigenstates, but rather are correlated states between the oscillator and the bath represented by the field  $\phi$ . The surprising feature here is that the bath is one with no energy gap. It is a massless field, which can carry off energy, and entropy or coherence at as low a frequency and energy as desired. There is nothing that is adiabatic as far as the  $\phi$  field is concerned.

### III. CORRELATIONS

In order to better understand these correlations between the oscillator and field, let us calculate them to see what they are. The field  $\phi$ , its conjugate momentum

$$\pi = \dot{\phi}(t, x) + \epsilon q(t) \delta(x) \quad (35)$$

commute with both the position  $q(t)$  and momentum  $p(t) = \dot{q}$  of the oscillator. Thus I will be interested in the correlation functions  $q(t)\phi(t', x)$  from which the equal time commutation relations can be calculated. In order to do so, we look at the correlators of the Fourier transforms

$$\langle q(\omega)\phi(\omega', x) \rangle \quad (36)$$

$$= \delta(\omega + \omega') \left[ i\epsilon\omega \frac{\langle (\phi_+(\omega, 0) + \phi_-(\omega, 0))(\phi^+(-\omega, x) + \phi_-(-\omega, x)) + i\gamma\omega(\phi_+(-\omega, 0) + \phi_-(-\omega, 0))e^{i\omega|x|} \rangle}{(\omega^2 - \Omega^2)^2 + \gamma^2\omega^2} \right] \quad (37)$$

$$= i\frac{\epsilon}{2\pi}\delta(\omega + \omega')\Theta(\omega) \left[ \frac{e^{i\omega|x|}}{(-\omega^2 + \Omega^2 - i\frac{\epsilon^2}{2}\omega)} + \frac{e^{-i\omega|x|}}{-\omega^2 + \Omega^2 + i\frac{\epsilon^2}{2}\omega} \right] \quad (38)$$

and

$$\langle \phi(\omega', x)q(\omega) \rangle = -\langle q(-\omega)\phi(-\omega', x) \rangle \quad (39)$$

This gives us that the commutator of  $q$  and  $\phi$  at equal times is

$$\langle q(t)\phi(t, x) - \phi(t, x)q(t) \rangle = 2i\frac{\epsilon}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{e^{i\omega|x|}}{(-\omega^2 + \Omega^2 - i\frac{\epsilon^2}{2}\omega)} \right] d\omega = 0 \quad (40)$$

as expected, and the correlation function is

$$\langle q(t)\phi(t, x) + \phi(t, x)q(t) \rangle = 4i\frac{\epsilon}{2\pi} \int_0^{\infty} \left[ \frac{e^{i\omega|x|}}{(-\omega^2 + \Omega^2 - i\frac{\epsilon^2}{2}\omega)} \right] d\omega \quad (41)$$

which can be written in terms of Exponential integrals. This correlation function falls off as  $e^{-\frac{\epsilon^2}{2}|x|}$ . That is, the correlation function falls off exponentially with distance from the oscillator, as one would expect from the exponential decay of the oscillator.

#### IV. CONCLUSION

The result we found was that although the density matrix of an oscillator strongly coupled to a scalar field leaves the oscillator in a state which is that of a thermal density matrix, the attempt to measure the temperature of that oscillator with a weakly coupled thermometer (in this case another oscillator weakly coupled to the first) produces a value of very near zero. This is surprising since the thermometer couples only to the first oscillator, which is in a mixed state. One would expect an object coupled to something in a mixed state to also be in a mixed state. However, that mixture comes about because of the coupling of the oscillator to another external heat bath which is in its ground state. If the thermometer were to instantaneously measure some aspect of the state of the oscillator it would find that oscillator to be a mixed state. All measurements made on it rapidly would produce results indistinguishable from the oscillator being in a thermal state with non-zero temperature. However, if the measurements are made slowly enough (over a time period long compared with the decay time of the oscillator), it is the heat bath and not the oscillator that dominate the dynamics, and measurement over that longer time period would find the oscillator to be in a pure state on average.

This is another example of false decoherence. While the oscillator looks to be decoherent over short time intervals, if one waits a long time, makes measurements averaged over a long time, that oscillator is in a pure state, the ground state.

This illustrates the dangers of using some simple measure of coherence or decoherence to make physical predictions. Decoherence is not some attribute that a body has, and thereafter retains. It is a property that depends both on its coupling to the external world, and on the time scale over which measurements are made.

Note that this result is not one about the ease of radiation emitted by a body. In the previous paper, the heat bath was a massive scalar field, and the energy of the oscillator was less than the mass of the scalar field. This meant that the system could not emit radiation at its natural frequency into the heat bath. It could still distort the field, so that the states of the field created by the different states of the system had a small overlap, and thus caused decoherence of the system when one traced out over the states of the field. However in the case discussed here, because of its coupling to a massless two dimensional field, the oscillator can radiate into that scalar field easily on any time scale. There is no inherent time scale associated with the scalar field. Furthermore, in the case of our thermometer, its dynamics is at its natural frequency, and at that frequency the scalar field has no problem supporting outgoing radiation, either energetically or in terms of phase space, in carrying away energy. Furthermore there is no question of Poincare recurrence or any other kind of recurrence. Once the scalar field is excited, all excitations are carried away at a fixed velocity, never to return. It is an infinite heat bath which accepts, and never returns, anything which is consigned to its depths. Nevertheless, the thermometer measures the oscillator to have zero temperature, to be in its ground state, if the coupling between the thermometer and the oscillator is small enough.

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- [1] See W. Unruh "False Loss of Coherence" in **Relativistic Quantum Measurement and Decoherence**, ed F. Petruccione, Springer Verlag(2000); A. J. Leggett in *Applications of Statistical and Field Theory Methods to Condensed Matter*(Proc. 1989 Nato Summer School, Evora, Portugal), ed D. Baeriswyl, A.R. Bishop, and J. Carmelo. Plenum Press (1990) and **Macroscopic Realism: What is it, and What do we know about it from Experiment** in *Quantum Measurement: Beyond Paradox* ed R. A. Healey, and G. Hellman, U. Minnesota Press (Minneapolis, 1998)